Supplemental Material for "Enhancing Nonreciprocity through Squeezing-Induced Symmetry Breaking"

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In this supplemental material, we first derive the effective Lindblad operator of the common reservoir. Then, we present the evolution equation of the operators within the squeezing framework. Considering the impact of the squeezed-cavity frequency, we conduct a systematic analysis. We further provide analytical expressions for the energy and ergotropy of the quantum battery, including a detailed illustration of the energy storage mechanism and the limitations that constrain the storage process. Finally, a detailed analysis was conducted on the scattering coefficient matrix of the optical isolator.

CONTENTS

I. DERIVATION OF THE EFFECTIVE LINDBLAD OPERATOR FOR THE COMMON RESERVOIR

Two bosonic modes coupled to a common reservoir exhibit the collective dissipative channel governed by the effective Lindblad operator $L_c = \sqrt{\Gamma}(p_a a + p_b b)$. We show that this mechanism can be implemented either via an auxiliary cavity mode that is adiabatically eliminated, or via directly coupling to a common one-dimensional waveguide.

A. Constructing L_c via the auxiliary cavity mode

We first consider two cavity modes a and b with the same frequency, interacting through a strongly damped auxiliary cavity mode z. In this configuration, the auxiliary mode decays rapidly and continually extracts information from

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the cavity modes a and b. Adiabatic elimination of this fast mode generates an effective Lindblad operator L_c. The detailed derivation is as follows. The Hamiltonian of the total system is

$$H^{\mathrm{tot}} = H + H^z + H^{\mathrm{int}}, \tag{S1}$$

with $H^z = \delta z^\dagger z$, $H^{\mathrm{int}} = (g_a a + g_b b)z^\dagger + \mathrm{H.c.} = Qz^\dagger + \mathrm{H.c.}$, where δ denotes the detuning between the cavity mode and the reservoir. The dynamics obeys the master equation

$$\dot{\rho}_{\mathrm{tot}} = -i[H^{\mathrm{tot}}, \rho_{\mathrm{tot}}] + \kappa_z \mathcal{L}[z]\rho_{\mathrm{tot}} = \mathcal{L}\rho_{\mathrm{tot}} + \mathcal{L}_z\rho_{\mathrm{tot}} + \mathcal{V}\rho_{\mathrm{tot}}, \tag{S2}$$

where $\mathcal{L}\bullet = -i[H, \bullet]$, $\mathcal{L}_z\bullet = -i[H^z, \bullet] + \kappa_z\mathcal{L}[z]\bullet$, and $\mathcal{V}\bullet = -i[H^{\mathrm{int}}, \bullet]$, with $\mathcal{L}[o]\rho = o\rho o^\dagger - 1/2\{o^\dagger o, \rho\}$. Because the auxiliary mode is strongly dissipative, $\kappa_z \gg \{g_a, g_b, \delta, ||H||\}$, the evolution of the subsystem governed by H is relatively slow, whereas the sector described by H^z relaxes rapidly. The auxiliary cavity relaxes to its unique steady state $\rho_z^{\mathrm{ss}} = |0\rangle_z\langle 0|$, which satisfies $\mathcal{L}_z\rho_z^{\mathrm{ss}} = 0$. To eliminate the fast subsystem, we employ the standard adiabatic–elimination expansion $\dot{\rho} = \sum_{i\geq 0} L_i\rho$, where $\rho = \mathrm{Tr}_z(\rho_{\mathrm{tot}})$ represents the density operator of the system composed of cavity modes a and b [S1–S3]. The zeroth-order contribution vanishes because the fast sector remains in ρ_z^{ss}, with $L_0\rho = 0$ [S4]. Under the first-order approximation, we obtain [S1]

$$L_1\rho = \mathcal{L}\rho + \mathrm{Tr}_z(\mathcal{V}\rho_{\mathrm{tot}}) = \mathcal{L}\rho - i\mathrm{Tr}_z([H^{\mathrm{int}}, \rho \otimes \rho_z^{\mathrm{ss}}]) = \mathcal{L}\rho. \tag{S3}$$

Furthermore, the dynamics of the slow subsystem can be represented by a linear, time-invariant Kraus map $\mathcal{K}$, such that the full density matrix satisfies $\rho_{\mathrm{tot}} = \mathcal{K}(\rho)$. Within the adiabatic-elimination expansion, the second-order contribution to the effective generator is given by [S1]

$$L_2\rho = \mathrm{Tr}_z(\mathcal{V}(\mathcal{K}_1(\rho))), \tag{S4}$$

with $\mathcal{K}_1(\rho) = \overline{\tau}\overline{\mathcal{K}}_z(\mathcal{V}(\rho \otimes \rho_z^{\mathrm{ss}})) + \mathbf{G}_{1,s}(\rho)$. Here $\mathbf{G}_{1,s}$ is the gauge choice associated with the completely-positive, trace-preserving representation for the reduced slow dynamics, where we select $\mathbf{G}_{1,s}(\rho) = -\overline{\tau}\mathrm{Tr}_z(\mathcal{V}(\rho \otimes \rho_z^{\mathrm{ss}}))$ for simplicity [S1, S5]. With this choice, we have

$$\begin{aligned}
\mathcal{K}_1(\rho) &= \overline{\tau}\overline{\mathcal{K}}_z(\mathcal{V}(\rho \otimes \rho_z^{\mathrm{ss}})) - \overline{\tau}\mathrm{Tr}_z(\mathcal{V}(\rho \otimes \rho_z^{\mathrm{ss}})) \\
&= -i\overline{\tau}\overline{\mathcal{K}}_z(Q\rho \otimes |1\rangle_z\langle 0| - \rho Q^\dagger \otimes |0\rangle_z\langle 1|) + i\overline{\tau}\mathrm{Tr}_z(Q\rho \otimes |1\rangle_z\langle 0| - \rho Q^\dagger \otimes |0\rangle_z\langle 1|) \\
&= -i\Big(\frac{Q\rho}{i\delta + \kappa_z/2} \otimes |1\rangle_z\langle 0| - \frac{\rho Q^\dagger}{-i\delta + \kappa_z/2} \otimes |0\rangle_z\langle 1|\Big).
\end{aligned} \tag{S5}$$

Then, the second–order contribution

$$\begin{aligned}
L_2\rho &= \mathrm{Tr}_z(\mathcal{V}(\mathcal{K}_1(\rho))) = -i\mathrm{Tr}_z([H^{\mathrm{int}}, \mathcal{K}_1(\rho)]) \\
&= -\frac{Q^\dagger Q\rho}{i\delta + \kappa_z/2} + \frac{Q\rho Q^\dagger}{-i\delta + \kappa_z/2} + \frac{Q\rho Q^\dagger}{i\delta + \kappa_z/2} - \frac{\rho Q^\dagger Q}{-i\delta + \kappa_z/2} \\
&= 2\alpha\big(Q\rho Q^\dagger - \frac{1}{2}\{Q^\dagger Q, \rho\}\big) + i[\delta H, \rho],
\end{aligned} \tag{S6}$$

where $\alpha = \kappa_z(2\delta^2 + \kappa_z^2/2)^{-1}$, $\delta H = \beta Q^\dagger Q$, and $\beta = \delta[\delta^2 + (\kappa_z/2)^2]^{-1}$.

Combining the zeroth-, first-, and second-order contributions in the adiabatic–elimination expansion, we obtain the effective master equation governing the reduced slow subsystem,

$$\dot{\rho} = -i[H - \delta H, \rho] + \Gamma_{\mathrm{eff}}\mathcal{L}[Q]\rho, \tag{S7}$$

with $\Gamma_{\mathrm{eff}} = 2\alpha$.

When the auxiliary mode is resonant with the system, one has $\beta = 0$, and therefore $\delta H = 0$. In this case, the effective dissipative rate reduces to $\Gamma_{\mathrm{eff}} = 4/\kappa_z$. Moreover, the operator Q can be written as $Q = g(p_a a + p_b b)$. As long as both coupling coefficients g_a and g_b are nonzero, they can be rescaled so that $|p_a||p_b| = 1$, with the overall magnitude absorbed into the coupling constant g. Consequently, the dissipative rate used in the main text is $\Gamma = g\sqrt{\Gamma_{\mathrm{eff}}}$, and the effective Lindblad operator is $L_c = \sqrt{\Gamma}(p_a a + p_b b)$.

B. Constructing L_c via a common waveguide

We demonstrate that directly coupling two cavity modes to a common 1D waveguide also provides a route to construct the collective dissipation operator L_c. Considering two cavity modes a and b at positions $R_a = -d/2$ and

$R_b = d/2$, the annihilation operators for the left- and right-propagating modes in the waveguide are denoted by $z_{k,L}$ and $z_{k,R}$. The total Hamiltonian is

$$H^{\text{tot}} = H + H^z + H^{\text{int}}, \tag{S8}$$

where the waveguide Hamiltonian is

$$H^z = \sum_k \nu_k (z_{k,L}^\dagger z_{k,L} + z_{k,R}^\dagger z_{k,R}), \tag{S9}$$

with k denotes the wave vector of the propagating mode and ν_k is the corresponding mode frequency. The interaction Hamiltonian is

$$H^{\text{int}} = \sum_{h=a,b} \sum_k \frac{g_k^h}{\sqrt{2}} (h^\dagger z_{k,R} e^{-i\nu_k t} + h z_{k,R}^\dagger e^{i\nu_k t} + h^\dagger z_{k,L} e^{i\nu_k t} + h z_{k,L}^\dagger e^{-i\nu_k t}). \tag{S10}$$

The evolution of the operator O (cavity modes subspace) in the Heisenberg picture is given by

$$\frac{dO}{dt} = i[H,O] + i \sum_{h=a,b} \sqrt{\frac{\Gamma_h}{2}} ([h^\dagger, O] z_{r_h,R} + [h,O] z_{r_h,R}^\dagger + [h^\dagger, O] z_{r_h,L} + [h,O] z_{r_h,L}^\dagger). \tag{S11}$$

In deriving the dissipative terms, we have used

$$\begin{aligned}\sum_k |g_k|^2 \int e^{-i(\omega-\nu_k)(t-\tau)} d\tau &= \sum_k 2\pi |g_k|^2 \delta(\omega-\nu_k) = L/2\pi \int dk 2\pi |g_k|^2 \delta(\omega-\nu_k) \\ &= L/2\pi \int dk 2\pi |g_k|^2 \delta(\omega-\nu_k) = |g_0|^2/v_g,\end{aligned} \tag{S12}$$

which yields the decay rate $\Gamma_h = |g_0^h|^2/v_g$ under the standard normalization $g_k^h = g_0^h/\sqrt{L}$. Here, $v_g = d\nu_k/dk$ denotes the group velocity of the transmission mode. Under the short-delay (Markovian) approximation, where retardation effects are neglected, the free-evolution approximation $h(t-\tau) \approx h(t)e^{i\omega_h \tau} = h(t)e^{ik_h d}$, where $\tau = d/v_g$ is the propagation delay associated with the cavity separation d, and $k_h = \omega_h/v_g$. Together with the slowly-varying decay-rate approximation $\Gamma_h(t-\tau) \approx \Gamma_h(t)$, and the assumption of vacuum input fields for the waveguide. The above formula can be further expressed as [S6]

$$\frac{dO}{dt} = i[H,O] + \sum_{h=a,b} \frac{\Gamma_h}{2} ([h^\dagger, O]h - [h,O]h^\dagger) + \sum_{h \neq h'=a,b} \frac{\sqrt{\Gamma_a \Gamma_b}}{2} ([h^\dagger, O]h' e^{ik_{h'}d} - h'^\dagger [h,O] e^{-ik_{h'}d}). \tag{S13}$$

Accordingly, the expectation value of O can be obtained. Since the above relation holds for an arbitrary operator O, the corresponding equation of motion for the density matrix can be directly identified, yielding the master equation

$$\frac{d\rho}{dt} = -i[H + H_J, \rho] + \sum_{h,h'=a,b} \Gamma_{hh'} \mathcal{D}(h,h')\rho, \tag{S14}$$

where the waveguide-mediated exchange interaction between the two cavities is $H_J = J' a^\dagger b + J'^* b^\dagger a$, with $J' = -i\sqrt{\Gamma_a \Gamma_b}/4(e^{ik_b d} - e^{-ik_a d})$. Here, $\Gamma_{hh} = \Gamma_h$ denotes the individual decay rate, and $\Gamma_{ab} = \Gamma_{ba}^* = \sqrt{\Gamma_a \Gamma_b}(e^{ik_b d} + e^{-ik_a d})/2$ characterizes the cooperative dissipation. The superoperator is defined as $\mathcal{D}(A,B)\rho = B\rho A^\dagger - 1/2\{A^\dagger B, \rho\}$. When the separation between the two cavities satisfies $d = 2n\pi/k$, with $n \in \mathbb{Z}$, the coherent interaction vanishes, $H_J = 0$, while the collective dissipative coupling $L_c = \sqrt{\Gamma}(p_a a + p_b b)$ is constructed,

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}[L_c]\rho, \tag{S15}$$

where we have set $k_a = k_b = k$, $\Gamma_a = \Gamma_b$.

II. EVOLUTION OF THE OPERATORS IN THE SQUEEZING FRAMEWORK

In this section, we consider a general scenario where the cavity modes a and b are squeezed with parameters (r_a, θ_a) and (r_b, θ_b), respectively, and both are coupled to a common squeezed reservoir characterized by (r_c, θ_c).

We first present the derivation of the master equation describing two cavity modes collectively coupled to a squeezing reservoir, using the waveguide as a concrete example. Other interactions are omitted for clarity and discussed later. The interaction Hamiltonian of the cavity modes with the reservoir is

$$H_{\rm int} = \sum_k \big[\frac{g_k^a}{\sqrt{2}}(a^\dagger z_{k,R}e^{i(\omega_a-\nu_k)t} + a^\dagger z_{k,L}e^{i(\omega_a+\nu_k)t}) + \frac{g_k^b}{\sqrt{2}}(b^\dagger z_{k,R}e^{i(\omega_b-\nu_k)t} + b^\dagger z_{k,L}e^{i(\omega_b+\nu_k)t})\big] + \text{H.c.}, \tag{S16}$$

within the Born–Markov approximation, the dynamics of the system is governed by

$$\dot\rho(t) = -i{\rm Tr}_z[H_{\rm int}(t), \rho(0)\otimes\rho_z(0)] - \int_0^t d\tau {\rm Tr}_z[H_{\rm int}(t), [H_{\rm int}(\tau), \rho(\tau)\otimes\rho_z(0)]], \tag{S17}$$

where $\rho_z(0)$ denotes the initial state of the reservoir.

For the squeezed vacuum reservoir, the density operator is given by

$$\rho_z = |\xi\rangle\langle\xi| = \prod_k S_k(\xi)|0_k\rangle\langle 0_k|S_k^\dagger(\xi), \tag{S18}$$

where the squeeze operator is

$$S_k(\xi) = \exp\big(\frac{1}{2}\xi^* z_{k_0+k}z_{k_0-k} - \frac{1}{2}\xi z^\dagger_{k_0+k}z^\dagger_{k_0-k}\big), \tag{S19}$$

with $\xi = r_c\exp(i\theta_c)$. The correlations can then be obtained:

$$\begin{aligned}
\langle z_k\rangle &= \prod_q \langle 0_q|S_q^\dagger z_k S_q|0_q\rangle = 0,\\
\langle z_k^\dagger\rangle &= \prod_q \langle 0_q|S_q^\dagger z_k^\dagger S_q|0_q\rangle = 0,\\
\langle z_k^\dagger z_{k'}\rangle &= \prod_q \langle 0_q|S_q^\dagger z_k^\dagger S_q S_q^\dagger z_{k'} S_q|0_q\rangle = N\delta_{kk'},\\
\langle z_k z_{k'}^\dagger\rangle &= \prod_q \langle 0_q|S_q^\dagger z_k S_q S_q^\dagger z_{k'}^\dagger S_q|0_q\rangle = (N+1)\delta_{kk'},\\
\langle z_k z_{k'}\rangle &= \prod_q \langle 0_q|S_q^\dagger z_k S_q S_q^\dagger z_{k'} S_q|0_q\rangle = -M^*\delta_{k',2k_0-k},\\
\langle z_k^\dagger z_{k'}^\dagger\rangle &= \prod_q \langle 0_q|S_q^\dagger z_k^\dagger S_q S_q^\dagger z_{k'}^\dagger S_q|0_q\rangle = -M\delta_{k',2k_0-k},
\end{aligned} \tag{S20}$$

where $N = \sinh^2 r_c$, $M = e^{-i\theta_c}\sinh r_c\cosh r_c$, and $k_0 = \omega_0/v_g$, with ω_0 is the central frequency of the squeezing device. These correlations are satisfied for both the left- and right-propagating modes.

Substituting $H_{\rm int}$ into Eq. (S17) and using the aforementioned correlation relationship, we derive the master equation in the squeezed framework

$$\begin{aligned}
\dot\rho =&\Gamma_a N(a^\dagger\rho a - 1/2\{aa^\dagger,\rho\}) + \Gamma_a(N+1)(a\rho a^\dagger - 1/2\{a^\dagger a,\rho\}) - \Gamma_a M(a\rho a - 1/2\{aa,\rho\})\\
&- \Gamma_a M^*(a^\dagger\rho a^\dagger - 1/2\{a^\dagger a^\dagger,\rho\}) + \Gamma_b N(b^\dagger\rho b - 1/2\{bb^\dagger,\rho\}) + \Gamma_b(N+1)(b\rho b^\dagger - 1/2\{b^\dagger b,\rho\})\\
&- \Gamma_b M(b\rho b - 1/2\{bb,\rho\}) - \Gamma_b M^*(b^\dagger\rho b^\dagger - 1/2\{b^\dagger b^\dagger,\rho\}) + \Gamma_{ab}N(a^\dagger\rho b - 1/2\{ba^\dagger,\rho\} + b^\dagger\rho a\\
&- 1/2\{ab^\dagger,\rho\}) + \Gamma_{ab}(N+1)(a\rho b^\dagger - 1/2\{b^\dagger a,\rho\} + b\rho a^\dagger - 1/2\{a^\dagger b,\rho\}) - \Gamma_{ab}M(a\rho b - 1/2\{ba,\rho\}\\
&+ b\rho a - 1/2\{ab,\rho\}) - \Gamma_{ab}M^*(a^\dagger\rho b^\dagger - 1/2\{b^\dagger a^\dagger,\rho\} + b^\dagger\rho a^\dagger - 1/2\{a^\dagger b^\dagger,\rho\}),
\end{aligned} \tag{S21}$$

with dissipation rate $\Gamma_h = |g_0^h|^2/v_g$, $\Gamma_{ab} = \sqrt{\Gamma_a\Gamma_b}$. Assuming $\Gamma_a = \Gamma_b$, this expression can be written as

$$\dot\rho = (N+1)\mathcal{L}[L_c]\rho + N\mathcal{L}[L_c^\dagger]\rho - M\mathcal{L}'[L_c]\rho - M^*\mathcal{L}'[L_c^\dagger]\rho. \tag{S22}$$

Furthermore, the Hamiltonian of the system in the squeezing framework is

$$H_s = S_b^\dagger S_a^\dagger(H + H_{\rm NL}^a + H_{\rm NL}^b)S_a S_b = \omega_s(a_s^\dagger a_s + b_s^\dagger b_s) + [Je^{i\varphi}(\cosh r_a a_s^\dagger - e^{i\theta_a}\sinh r_a a_s)(\cosh r_b b_s - e^{-i\theta_b}\sinh r_b b_s^\dagger) + \text{H.c.}],$$

including the local dissipation of the cavity modes, the full master equation can be written as

$$
\begin{aligned}
\dot{\rho_s}(t) = &- i[H_s,\rho_s(t)] + \sum_{j=a,b} S_b^\dagger S_a^\dagger \mathcal{L}[L_j]\rho(t)S_aS_b + (N+1)S_b^\dagger S_a^\dagger \mathcal{L}[L_c]\rho(t)S_aS_b + NS_b^\dagger S_a^\dagger \mathcal{L}[L_c^\dagger]\rho(t)S_aS_b \\
&- MS_b^\dagger S_a^\dagger \mathcal{L}'[L_c]\rho(t)S_aS_b - M^* S_b^\dagger S_a^\dagger \mathcal{L}'[L_c^\dagger]\rho(t)S_aS_b,
\end{aligned}
$$

which corresponding to Eq. (1) in the main text. The evolution of the operators is given by the following expressions:

$$
\begin{aligned}
\frac{d\langle a_s\rangle}{dt} =& -\frac{\Lambda_a+i\omega_s}{2}\langle a_s\rangle - (\zeta^* e^{-i\theta_a}\sinh r_a\cosh r_b - \zeta e^{-i\theta_b}\sinh r_b\cosh r_a)\langle b_s^\dagger\rangle + (\zeta^* e^{-i\Delta\theta}\sinh r_a\sinh r_b \\
&- \zeta\cosh r_a\cosh r_b)\langle b_s\rangle, \\
\frac{d\langle b_s\rangle}{dt} =& -\frac{\Lambda_b+i\omega_s}{2}\langle b_s\rangle + (\eta e^{-i\theta_a}\sinh r_a\cosh r_b - \eta^* e^{-i\theta_b}\sinh r_b\cosh r_a)\langle a_s^\dagger\rangle + (\eta^* e^{i\Delta\theta}\sinh r_a\sinh r_b \\
&- \eta\cosh r_a\cosh r_b)\langle a_s\rangle, \\
\frac{d\langle a_s^\dagger a_s\rangle}{dt} =& -\Lambda_a\langle a_s^\dagger a_s\rangle + (\zeta e^{-i\theta_b}\sinh r_b\cosh r_a - \zeta^* e^{-i\theta_a}\sinh r_a\cosh r_b)\langle a_s^\dagger b_s^\dagger\rangle + (\zeta^* e^{i\theta_b}\sinh r_b\cosh r_a - \zeta e^{i\theta_a} \\
&\times\sinh r_a\cosh r_b)\langle a_s b_s\rangle - (\zeta\cosh r_a\cosh r_b - \zeta^* e^{-i\Delta\theta}\sinh r_a\sinh r_b)\langle a_s^\dagger b_s\rangle + (\zeta e^{i\Delta\theta}\sinh r_a\sinh r_b \\
&- \zeta^*\cosh r_a\cosh r_b)\langle a_s b_s^\dagger\rangle + \Gamma[p_a^2 e^{-i(\theta_c+\theta_a)} + p_a^{*2}e^{i(\theta_c+\theta_a)}]\sinh r_a\cosh r_a\sinh r_c\cosh r_c + \Gamma|p_a|^2 \\
&\times\sinh^2 r_a\cosh^2 r_c + \Gamma|p_a|^2\cosh^2 r_a\sinh^2 r_c, \\
\frac{d\langle a_s b_s\rangle}{dt} =& -\frac{\Lambda_a+\Lambda_b+4i\omega_s}{2}\langle a_s b_s\rangle + (\eta e^{-i\theta_a}\sinh r_a\cosh r_b - \eta^* e^{-i\theta_b}\sinh r_b\cosh r_a)\langle a_s^\dagger a_s\rangle + (-\zeta^* e^{-i\theta_a}\sinh r_a \\
&\times\cosh r_b + \zeta e^{-i\theta_b}\sinh r_b\cosh r_a)\langle b_s^\dagger b_s\rangle - (\zeta\cosh r_a\cosh r_b - \zeta^* e^{-i\Delta\theta}\sinh r_a\sinh r_b)\langle b_s b_s\rangle + (\eta^* e^{i\Delta\theta} \\
&\times\sinh r_a\sinh r_b - \eta\cosh r_a\cosh r_b)\langle a_s a_s\rangle + iJe^{-i(\varphi+\theta_a)}\sinh r_a\cosh r_b + iJe^{i(\varphi-\theta_b)}\sinh r_b\cosh r_a \\
&+ \frac{\Gamma}{2}p_a p_b e^{-i(\theta_c+\theta_a+\theta_b)}\sinh r_a\sinh r_b\sinh 2r_c + \frac{\Gamma}{2}\mu^* e^{-i\theta_a}\sinh r_a\cosh r_b\cosh 2r_c + \frac{\Gamma}{2}p_a^* p_b^* e^{i\theta_c}\cosh r_a \\
&\times\cosh r_b\sinh 2r_c + \frac{\Gamma}{2}\mu e^{-i\theta_b}\sinh r_b\cosh r_a\cosh 2r_c, \\
\frac{d\langle a_s^\dagger b_s\rangle}{dt} =& -\frac{\Lambda_a+\Lambda_b}{2}\langle a_s^\dagger b_s\rangle + (\zeta^* e^{i\theta_b}\sinh r_b\cosh r_a - \zeta e^{i\theta_a}\sinh r_a\cosh r_b)\langle b_s b_s\rangle + (\eta e^{-i\theta_a}\sinh r_a\cosh r_b - \eta^* \\
&\times e^{-i\theta_b}\sinh r_b\cosh r_a)\langle a_s^\dagger a_s^\dagger\rangle + (\eta^* e^{i\Delta\theta}\sinh r_a\sinh r_b - \eta\cosh r_a\cosh r_b)\langle a_s^\dagger a_s\rangle + (\zeta e^{i\Delta\theta}\sinh r_a \\
&\times\sinh r_b - \zeta^*\cosh r_a\cosh r_b)\langle b_s^\dagger b_s\rangle + \Gamma p_a p_b e^{-i(\theta_b+\theta_c)}\sinh r_b\cosh r_a\sinh r_c\cosh r_c + \Gamma p_a^* p_b^* e^{i(\theta_a+\theta_c)} \\
&\times\sinh r_a\cosh r_b\sinh r_c\cosh r_c + \mu\Gamma e^{i\Delta\theta}\sinh r_a\sinh r_b\cosh^2 r_c + \mu^*\Gamma\cosh r_a\cosh r_b\sinh^2 r_c, \\
\frac{d\langle a_s a_s\rangle}{dt} =& -(\Lambda_a+2i\omega_s)\langle a_s a_s\rangle - 2(\zeta^* e^{-i\theta_a}\sinh r_a\cosh r_b - \zeta e^{-i\theta_b}\sinh r_b\cosh r_a)\langle a_s b_s^\dagger\rangle - 2(\zeta\cosh r_a\cosh r_b \\
&+ \zeta^* e^{-i\Delta\theta}\sinh r_a\sinh r_b)\langle a_s b_s\rangle + \Gamma p_a^2 e^{-i(2\theta_a+\theta_c)}\sinh^2 r_a\sinh r_c\cosh r_c + \Gamma p_a^{*2}e^{i\theta_c}\cosh^2 r_a\sinh r_c \\
&\times\cosh r_c + \Gamma|p_a|^2 e^{-i\theta_a}\sinh r_a\cosh r_a\cosh 2r_c, \\
\frac{d\langle b_s b_s\rangle}{dt} =& -(\Lambda_b+2i\omega_s)\langle b_s b_s\rangle + 2(\eta e^{-i\theta_a}\sinh r_a\cosh r_b - \eta^* e^{-i\theta_b}\sinh r_b\cosh r_a)\langle a_s^\dagger b_s\rangle - 2(\eta\cosh r_a\cosh r_b \\
&+ \eta^* e^{i\Delta\theta}\sinh r_a\sinh r_b)\langle a_s b_s\rangle + \Gamma p_b^2 e^{-i(2\theta_b+\theta_c)}\sinh^2 r_b\sinh r_c\cosh r_c + \Gamma p_b^{*2}e^{i\theta_c}\cosh^2 r_b\sinh r_c \\
&\times\cosh r_c + \Gamma|p_b|^2 e^{-i\theta_b}\sinh r_b\cosh r_b\cosh 2r_c, \\
\frac{d\langle b_s^\dagger b_s\rangle}{dt} =& -\Lambda_b\langle b_s^\dagger b_s\rangle + (\eta e^{-i\theta_a}\sinh r_a\cosh r_b - \eta^* e^{-i\theta_b}\sinh r_b\cosh r_a)\langle a_s^\dagger b_s^\dagger\rangle - (\eta e^{i\theta_b}\sinh r_b\cosh r_a - \eta^* e^{i\theta_a} \\
&\times\sinh r_a\cosh r_b)\langle a_s b_s\rangle - (\eta^*\cosh r_a\cosh r_b - \eta e^{-i\Delta\theta}\sinh r_a\sinh r_b)\langle a_s^\dagger b_s\rangle + (\eta^* e^{i\Delta\theta}\sinh r_a\sinh r_b \\
&- \eta\cosh r_a\cosh r_b)\langle a_s b_s^\dagger\rangle + \Gamma|p_b|^2\sinh^2 r_b\cosh^2 r_c + \Gamma|p_b|^2\cosh^2 r_b\sinh^2 r_c + \Gamma[p_b^2 e^{-i(\theta_c+\theta_b)} + p_b^{*2} \\
&\times e^{i(\theta_c+\theta_b)}]\sinh r_b\cosh r_b\sinh r_c\cosh r_c,
\end{aligned} \tag{S23}
$$

where $\Lambda_j = \Gamma|p_j|^2 + \kappa_j$, $\Delta\theta = \theta_a - \theta_b$, $\zeta = iJe^{i\varphi} + \mu\frac{\Gamma}{2}$, $\eta = iJe^{-i\varphi} + \mu^*\frac{\Gamma}{2}$, and $\mu = p_a^* p_b$.

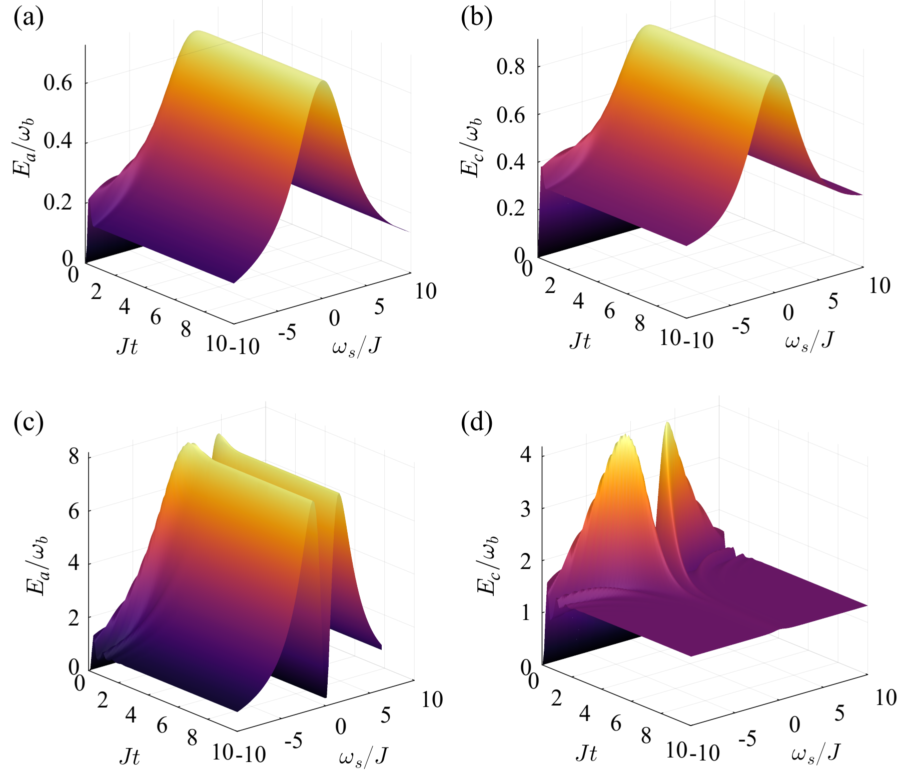

FIG. S1. Energy of quantum battery versus the scaled time Jt and squeezed-cavity frequency ω_s. (a,b) correspond to $J/\omega_b = 1 \times 10^{-5}$ for case (a) and case (c), respectively, while (c,d) correspond to $J/\omega_b = 0.001$. The parameters used are $\kappa = 8 \times 10^{-5}\omega_b$, $\epsilon = 10^{-4}\omega_b$, $r = 1$, and $\Gamma = 2J$.

III. CONSIDERATIONS ON THE SQUEEZED-CAVITY FREQUENCY FOR BATTERY ENERGY

For the quantum battery, a resonant classical drive field is applied to the charger, supplying energy to the system, which is described by the Hamiltonian $H_c = \epsilon(a + a^\dagger)$, where ϵ denotes the driving strength. After the squeezing transformation, the nonlinear Hamiltonian of the charger reduces to $H^a_{\rm NL} = \omega_s a_s^\dagger a_s$, where ω_s denotes the squeezed-cavity frequency, i.e., there exists a detuning between the cavity modes a_s and b. For cases (a) and (c), we show the energy as a function of ω_s in Fig. S1. For the Jaynes–Cummings model in the squeezing framework, the efficiency of the storage energy is determined by the detuning ω_s and coupling strength J. In the weak coupling regime, detuning suppresses the energy exchange between the two cavities, and the energy stored under resonance conditions is optimal, as shown in Figs. S1(a, b).

In contrast, in the strong coupling regime, due to the dissipation of the cavity, the resonance maximizes the exchange rate, but simultaneously enhances energy leakage through dissipative channels, thereby reducing the stored energy. In this case, a finite detuning can establish a balance between the exchange rate and dissipation, leading to the existence of an optimal detuning, as shown in Figs. S1(c, d). For analytical convenience of the derivations and calculations, we set $\omega_s = 0$ hereafter and in the main text.

IV. ENERGY OF THE SQUEEZING-IMPROVED NONRECIPROCAL QUANTUM BATTERY

In this section, we first solve the equations of motion above to obtain analytical expressions for the battery energy under different cases, and then provide a detailed analysis of the underlying energy storage mechanism and the applicability of the model. For the case (a), the stored energy is given by

$$E_a = \omega_b\langle b^\dagger b\rangle = -\frac{4\omega_b J^2 e^{-2r-(2\Lambda_a+\frac{3\Lambda_b}{2})t}}{\Lambda_a^2\Lambda_b^2\Lambda_-^2\Lambda_+}\Big\{(1+e^{4r})(2J-\Lambda_a)\big[\Lambda_a\Lambda_b\Lambda_-^2 e^{(2\Lambda_a+\frac{3\Lambda_b}{2})t} + 4\Lambda_a^2\Lambda_b^2 e^{(\Lambda_b+\frac{3\Lambda_a}{2})t}\big] + 32\epsilon^2$$
$$\times\, \Lambda_+\Lambda_- e^{2r}\big[\Lambda_a e^{(2\Lambda_a+\Lambda_b)t} - \Lambda_b e^{\frac{3}{2}\Lambda_+ t}\big] + \Lambda_a\Lambda_b\Lambda_+(1+e^{4r})(\Lambda_a-2J)\big[\Lambda_a e^{(2\Lambda_a+\frac{1}{2}\Lambda_b)t} + \Lambda_b e^{(\Lambda_a+\frac{3}{2}\Lambda_b)t}\big] - 2$$

$$\times e^{2r+(\Lambda_a+\frac{3\Lambda_b}{2})t}\Lambda_b^2\Lambda_+(8\epsilon^2-2J\Lambda_a+\Lambda_a^2)-2e^{2r+(2\Lambda_a+\frac{\Lambda_b}{2})t}\Lambda_a^2\Lambda_+(8\epsilon^2-2J\Lambda_b+\Lambda_a\Lambda_b)+8e^{2r+(\Lambda_b+\frac{3\Lambda_a}{2})t}$$
$$\times\Lambda_a\Lambda_b(4\epsilon^2\Lambda_+-2J\Lambda_a\Lambda_b+\Lambda_a^2\Lambda_b)-2\Lambda_-^2e^{2r+(2\Lambda_a+\frac{3\Lambda_b}{2})t}(8\epsilon^2\Lambda_++2J\Lambda_a\Lambda_b-\Lambda_a^2\Lambda_b)\Big\}, \tag{S24}$$

where $\Lambda_\pm=\Lambda_a\pm\Lambda_b$, we have set $\omega_s=0$ and $r_a=r_c=r$ for simplicity.

For case (b), the stored energy is

$$E_b=\frac{J\omega_b e^{-(\frac{3\Lambda_a}{2}+\Lambda_b)t}}{\Lambda_a^2\Lambda_b^2\Lambda_-^2\Lambda_+}\Big\{128J\epsilon^2\Lambda_+\Lambda_-\big[\Lambda_b e^{\Lambda_+t}-\Lambda_a e^{\frac{1}{2}(3\Lambda_a+\Lambda_b)t}\big]+16J\Lambda_b^2\Lambda_+e^{\frac{1}{2}t(\Lambda_a+2\Lambda_b)}(4\epsilon^2+J\Lambda_a)+\Lambda_a^2\Lambda_+e^{\frac{3t\Lambda_a}{2}}$$
$$\times\big[64J\epsilon^2+(4J-\Lambda_a)^2\Lambda_b+2(4J-\Lambda_a)\Lambda_b^2+\Lambda_b^3\big]+16e^{\frac{1}{2}(2\Lambda_a+\Lambda_b)t}J\Lambda_a\Lambda_b\big[\Lambda_a\Lambda_b(\Lambda_--4J)-8\epsilon^2\Lambda_+\big]-\Lambda_-^2$$
$$\times e^{(\frac{3}{2}\Lambda_a+\Lambda_b)t}(16J^2\Lambda_a\Lambda_b+\Lambda_a^2\Lambda_b\Lambda_+-8J\Lambda_a^2\Lambda_b-64J\epsilon^2\Lambda_+)-\Lambda_a\Lambda_b\big[16Je^{\frac{1}{2}(2\Lambda_a+\Lambda_b)t}\Lambda_a\Lambda_b(\Lambda_--4J)+16J^2$$
$$\times\Lambda_b\Lambda_+e^{\frac{1}{2}(\Lambda_a+2\Lambda_b)t}+e^{\frac{3}{2}\Lambda_a t}\Lambda_a\Lambda_+(4J-\Lambda_-)^2\big]-\Lambda_a\Lambda_b\Lambda_-^2e^{(\frac{3}{2}\Lambda_a+\Lambda_b)t}\big[(-4J+\Lambda_a)^2+\Lambda_a\Lambda_b\big]\cosh 2r\Big\}. \tag{S25}$$

And for case (c), the energy is

$$E_c=\frac{e^{-\Lambda_b t}}{2\Lambda_a^2\Lambda_b^2\Lambda_-^2\Lambda_+}\Big\{\Lambda_+\big[(1-e^{\Lambda_b t})\Gamma\Lambda_a^2\Lambda_-^2\Lambda_b+16J^2(e^{-\frac{1}{2}\Lambda_-t}-1)^2\Lambda_a^2\Lambda_b^2+128J^2e^{-\Lambda_a t}\epsilon^2(\Lambda_a e^{\frac{1}{2}\Lambda_a t}-\Lambda_b e^{\frac{1}{2}\Lambda_b t}$$
$$-\Lambda_-\Lambda_a e^{\frac{1}{2}\Lambda_+t})^2\big]+\Lambda_a\Lambda_b\big[(e^{\Lambda_b t}-1)\Gamma\Lambda_a\Lambda_-^2\Lambda_++8J^3(\Lambda_a\Lambda_+-\Lambda_-^2e^{\Lambda_b t}-4\Lambda_a\Lambda_b e^{-\frac{1}{2}\Lambda_-t}+\Lambda_b\Lambda_+e^{-\Lambda_-t})$$
$$-16J^2(e^{-\frac{1}{2}\Lambda_-t}-1)^2\Lambda_a\Lambda_b\Lambda_++4J^2\Gamma(\Lambda_-^2e^{\Lambda_b t}+4\Lambda_a\Lambda_b e^{-\frac{1}{2}\Lambda_-t}-\Lambda_a^2-\Lambda_b\Lambda_+e^{-\Lambda_-t})\big]\cosh 2r+4J^2\Lambda_a$$
$$\times\Lambda_b(2J-\Gamma)(\Lambda_-^2e^{\Lambda_b t}+4\Lambda_a\Lambda_b e^{-\frac{1}{2}\Lambda_-t}-\Lambda_a\Lambda_+-\Lambda_b\Lambda_+e^{-\Lambda_-t})\cosh 6r\Big\}. \tag{S26}$$

A. Analysis of energy storage mechanisms

Dissipation drives the system to a steady state; the corresponding stored energy is given by Eq. (5) in the main text. Here, we present a detailed analysis of how squeezing the charger and the reservoir affects the battery energy. For the case of only squeezing the charger, the master equation of the system can be expanded as

$$\begin{aligned}\dot\rho_s(t)=&-i[H_s,\rho_s(t)]+\sum_{j=a,b}S_a^\dagger\mathcal{L}[L_j]\rho(t)S_a+S_a^\dagger\mathcal{L}[L_c]\rho(t)S_a\\=&-i[H_s,\rho_s(t)]+\sum_{j=a_s,b}\mathcal{L}[L_j]\rho_s(t)+\Gamma\big[a_s\rho_s(t)a_s^\dagger-1/2\{a_s^\dagger a_s,\rho_s(t)\}\big]+\Gamma\big[b\rho_s(t)b^\dagger-1/2\{b^\dagger b,\rho_s(t)\}\big]\\&+\Gamma p_ap_b^*\mathcal{D}(b,a_s)\rho_s(t)+\Gamma p_ap_b^*\mathcal{D}(a_s,b)\rho_s(t)\\=&-i[H_s,\rho_s(t)]+\sum_{j=a_s,b}\mathcal{L}[L_j']\rho_s(t)+\Gamma p_ap_b^*\mathcal{D}(b,a_s)\rho_s(t)+\Gamma p_a^*p_b\mathcal{D}(a_s,b)\rho_s(t),\end{aligned} \tag{S27}$$

where $\mathcal{D}(A,B)$ describes the cooperative dissipative coupling between the charger and battery, whose interplay with the coherent coupling gives rise to nonreciprocity. In addition, $L'_{a_s(b)}=\sqrt{\kappa_{a(b)}+\Gamma}a_s(b)$ indicates that the common reservoir also enhances the local dissipation rates. There exists an optimal $J_{\rm op}$ that maximizes the steady-state energy.

We note that squeezing the reservoir gives rise to a qualitatively different behavior. Here we provide a detailed analysis where only the common reservoir is squeezed. The pair correlations in the squeezed vacuum reservoir increase exponentially with the squeezing parameter, allowing the system to exchange excitations in correlated pairs, which does not modify the *first*-order dynamics, but instead reshapes the *higher*-order dynamics of the system.

Indeed, from the equations of motion derived in Eq. (S23), setting $r_a=r_b=0$ yields the following expressions for the *second*-order moments:

$$\begin{aligned}\frac{d\langle a^\dagger a\rangle}{dt}&=-\Lambda_a\langle a^\dagger a\rangle-\zeta\langle a^\dagger b\rangle-\zeta^*\langle ab^\dagger\rangle+i\epsilon\langle a\rangle-i\epsilon\langle a^\dagger\rangle+\Gamma|p_a|^2\sinh^2 r_c,\\\frac{d\langle a^\dagger b\rangle}{dt}&=-\frac{\Lambda_a+\Lambda_b}{2}\langle a^\dagger b\rangle-\eta\langle a^\dagger a\rangle-\zeta^*\langle b^\dagger b\rangle+i\epsilon\langle b\rangle+\mu^*\Gamma\sinh^2 r_c,\\\frac{d\langle b^\dagger b\rangle}{dt}&=-\Lambda_b\langle b^\dagger b\rangle-\eta^*\langle a^\dagger b\rangle-\eta\langle ab^\dagger\rangle+\Gamma|p_b|^2\sinh^2 r_c.\end{aligned} \tag{S28}$$

Squeezing the common reservoir introduces a constant gain term ($\Gamma|p_b|^2\sinh^2 r_c$) in the equation for $\langle b^\dagger b\rangle$, which originates from the effective occupied number $N=\sinh^2 r_c$ of the squeezed reservoir. Since this constant contribution

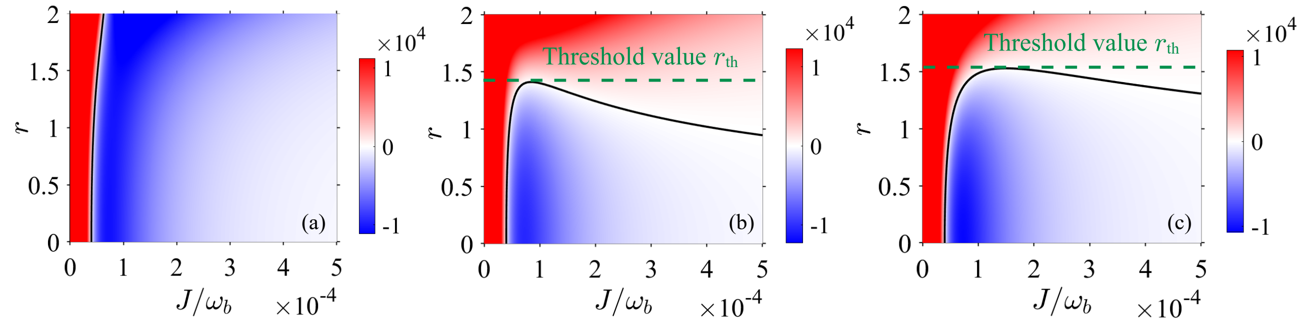

FIG. S2. Partial derivative $\partial E_i^{\rm ss}/\partial J$ of the steady-state energy $E_i^{\rm ss}$ versus the coherent coupling strength J and the squeezing parameter r for case (a), case (b), and case (c).

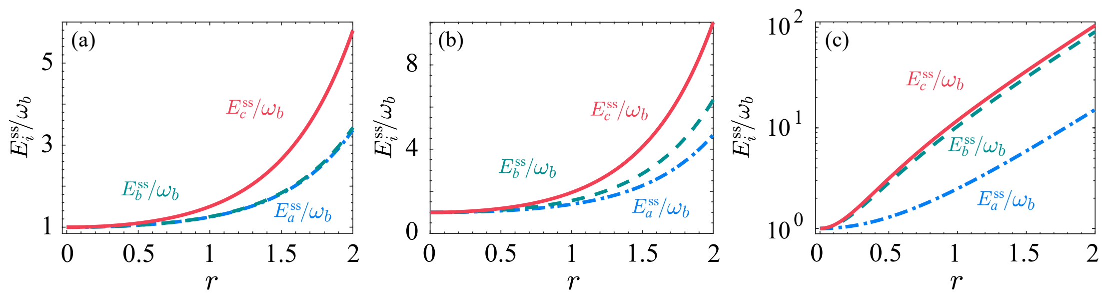

FIG. S3. Steady-state energy versus the squeezing parameter r for different coupling strengths, with $J/\omega_b = 5\times10^{-5}$ for (a), $J/\omega_b = 10^{-4}$ for (b), and $J/\omega_b = 5\times10^{-4}$ for (c).

exists independently of the buildup of coherence and correlations, it enhances the effective energy transfer rate into the battery, thereby significantly increasing the stored energy and enabling faster charging. But for the squeezing of the charger, the gain term affects the battery energy only indirectly through the coherent coupling between operators; the battery acquires energy only after the relevant coherence and correlations are established. Moreover, the squeezed reservoir not only increases system energy but also provides ordered quantum coherent resources. In particular, the anomalous correlation M introduces phase-sensitive quantum coherence that cannot be generated by a thermal reservoir. These coherence modify the system dynamics and enhance the extractable work within the system.

For cases (b) and (c), a threshold squeezing strength $r_{\rm th}$ emerges, above which the steady-state energy increases monotonically with the coupling strength J. This behavior originates from the fact that the dissipative channels inject energy into the system. As the squeezing strength increases, these contributions become dominant, masking the competition between coherent coupling and dissipative coupling. For all cases (a, b, c), the derivative of the energy takes the form

$$
\begin{aligned}
\frac{\partial E_a^{\rm ss}}{\partial J} &= \frac{16J[8\epsilon^2(-2J+\kappa)+\kappa(-J+\kappa)(2J+\kappa)\sinh^2 r]}{(2J+\kappa)^5},\\
\frac{\partial E_b^{\rm ss}}{\partial J} &= \frac{128J\epsilon^2(-2J+\kappa)+2\kappa(2J+\kappa)(12J^2-4J\kappa+\kappa^2)\sinh^2 r}{(2J+\kappa)^5},\\
\frac{\partial E_c^{\rm ss}}{\partial J} &= \frac{2[64J\epsilon^2(-2J+\kappa)+\kappa(2J+\kappa)^3\sinh^2 r]}{(2J+\kappa)^5}.
\end{aligned}
\tag{S29}
$$

As shown in Fig. S2, it remains zero over the entire parameter range when $r > r_{\rm th}$, indicating a strictly monotonic dependence of the steady-state energy on J.

Furthermore, the enhancement of nonreciprocal coupling represents only one aspect of the squeezing-induced effects. In the squeezed framework, additional counter-rotating contributions naturally arise and and the classical driving field is amplified to $\epsilon[(\cosh r_a - e^{i\theta_a}\sinh r_a)a_s + (\cosh r_a - e^{-i\theta_a}\sinh r_a)a_s^\dagger]$. Figure S3 shows that, in both the strong- and weak-coupling regimes, the stored energy increases monotonically with the squeezing strength r, reflecting the cooperative enhancement of effective energy-injection channels induced by squeezing.

B. Limitations of energy storage

According to Eq. (5) in the main text, the steady-state energies can be expressed as

$$E_i^{\rm ss} = A_i \sinh^2 r + E^{\rm ss}, \quad (i = a, b, \text{or } c),$$

with

$$A_a = \frac{8J^2\kappa}{(2J+\kappa)^3}, \quad A_b = \frac{2J(4J^2+\kappa^2)}{(2J+\kappa)^3}, \quad A_c = \frac{2J}{2J+\kappa}.$$

For the coherent coupling strength J, the energy stored in the battery exhibits an optimum coupling strength $J_{\rm op}$, at which the stored energy reaches a maximum [see the white dashed lines in Fig. 3(a-c) of the main text]. This can be understood from the competition between coherent energy exchange and dissipation, leading to an effective matching condition. In contrast, for the squeezing parameter, the derivative of the stored energy with respect to r is

$$\frac{\partial E_i^{\rm ss}}{\partial r} = A_i \sinh(2r) > 0,$$

which is always positive. The correlation coefficients of the squeezing vacuum reservoir are related to N and M, both of which increase monotonically and are unbounded with respect to the squeezing parameter. As a result, the squeezing-induced contributions in the master equation grow with r, and the corresponding stored energy does not exhibit saturation.

These conclusions are derived under the assumption of a broadband squeezed reservoir, where the reservoir correlation time is much shorter than all relevant system timescales. However, in realistic physics platforms, the stored energy may be limited by imperfections of the squeezing source and of the propagation/coupling channels, including finite pump power and pump depletion, optical or microwave losses, phase noise, excess thermal noise or heating, and the finite bandwidth of the engineered reservoir. In particular, loss and phase noise degrade the observable squeezing, while pump depletion causes the pump field to evolve dynamically, thereby modifying the reservoir correlations. Moreover, when the squeezing bandwidth is no longer sufficiently large compared with the relevant spectral scales of the system, reservoir temporal correlations cannot be neglected, and the broadband Markovian description no longer holds. In this regime, a more complicated model incorporating finite bandwidth and pump dynamics would be required. Alternatively, these effects can be mitigated via appropriate engineering techniques, allowing the system to still be described well within a Markovian framework. Exploring these directions remains worthy for future exploration.

V. ANALYTICAL CALCULATION FOR THE ERGOTROPY OF THE QUANTUM BATTERY

Apart from the energy storage performance, the extractable energy (ergotropy) is also an important criterion for evaluating batteries, defined as $\mathcal{E}_i^{\rm ss} = E_i^{\rm ss} - \widetilde{E}_i^{\rm ss} = {\rm Tr}[\rho_b(t)H_b] - {\rm Tr}[\widetilde{\rho}_b(t)H_b]$. Here, $E_i^{\rm ss}$ denotes the stored energy of the battery, $\widetilde{\rho}_b$ is the passive state, from which no work can be extracted via unitary cyclic processes. Explicitly, if $\rho_b = \sum_n r_n|r_n\rangle\langle r_n|$, with $r_1 \geq r_2 \geq \cdots$, and $H_b = \sum_n \epsilon_n|\epsilon_n\rangle\langle\epsilon_n|$ with $\epsilon_1 \leq \epsilon_2 \leq \cdots$, then $\widetilde{\rho}_b = \sum_n r_n|\epsilon_n\rangle\langle\epsilon_n|$. Our numerical results are obtained by solving the master equation and substituting the resulting density matrix into the above expression. For the solution of analytical expression, the corresponding passive energy $\widetilde{E}_i^{\rm ss}$ can be expressed as [S7, S8]

$$\widetilde{E}_i^{\rm ss} = \omega_b\left(\frac{\sqrt{\mathcal{J}_i}-1}{2}\right), \tag{S30}$$

with

$$\mathcal{J} = \left(1 + 2\langle b^\dagger b\rangle - 2\langle b^\dagger\rangle\langle b\rangle\right)^2 - 4\left|\langle bb\rangle - \langle b\rangle^2\right|^2.$$

By solving the equation of motion for the operators, the ergotropy for the battery can also be obtained; the corresponding $\mathcal{J}$ is

$$\begin{aligned}\mathcal{J}_a =& \frac{256J^4\sinh^2 r}{\Lambda^8}\Big\{\big[8(1-i)\epsilon^2 + (2J-\Lambda)\Lambda\big]\cosh r - 8(1-i)\epsilon^2\sinh r\Big\}\Big\{\big[-8(1+i)\epsilon^2 - (2J-\Lambda)\Lambda\big]\cosh r \\ &+ 8(1+i)\epsilon^2\sinh r\Big\} + \frac{1}{\Lambda^6}\big[-8J^2\Lambda + \Lambda^3 + 8J^2(\Lambda\cosh 2r - 4J\sinh^2 r)\big]^2,\end{aligned}$$

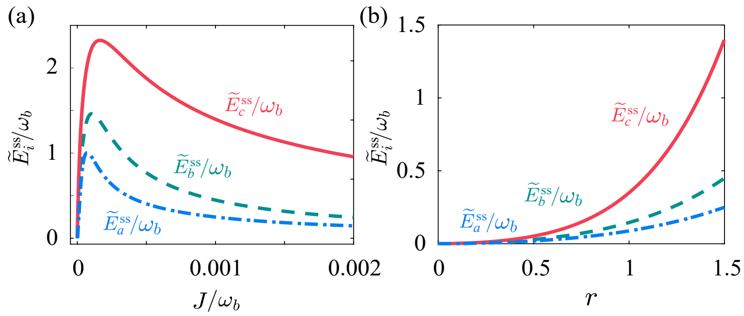

FIG. S4. Energy $\widetilde{E}_i^{ss}$ of the passive state as the function of the coupling strength J (a) and squeezing parameter r (b) in the steady state. The parameters used are $\kappa = 8\times10^{-5}\omega_b$, $\epsilon = 10^{-4}\omega_b$, $r = 1.5$, $J/\omega_b = 0.001$, and $\Gamma = 2J$.

$$\begin{aligned}\mathcal{J}_b =&\frac{1}{\Lambda^8}\Big\{\Lambda^2\big[(2J-\Lambda)(8J^2+\Lambda^2)-2J(8J^2-4J\Lambda+\Lambda^2)\cosh 2r\big]^2 - J^2\big[64J\epsilon^2(1-\cosh r)+(2\Lambda^3+16J^2\Lambda\\&-8J\Lambda^2)\sinh 2r\big]^2\Big\},\\ \mathcal{J}_c =&\frac{1}{\Lambda^6}\Big\{4J\Lambda^4(\Lambda-2J)\cosh 2r-(8J^2-4J\Lambda+\Lambda^2)\big[-32J^4+16J^3\Lambda-\Lambda^4+16J^3(2J-\Lambda)\cosh 4r\big]\Big\},\end{aligned} \tag{S31}$$

where we have set $\Lambda_a = \Lambda_b = \Lambda$ for simplicity. Substituting it into Eq. (S30), it is easy to obtain the analytical solution of $\mathcal{E}_i^{ss}$ for different cases.

The energy of the passive state is illustrated in Fig. S4. At the optimal coupling strength J, the corresponding $\widetilde{E}_i^{ss}$ also attains its maximal value. Moreover, $\widetilde{E}_i^{ss}$ increases monotonically with the squeezing parameter r. This result indicates that the increase in steady-state energy does not necessarily lead to an increase in ergotropy. Squeezing of the reservoir can stably maintain coherence in the steady state, maximizing the ergotropy of the battery; while the additional battery squeezing increases energy, it mainly introduces a passive state, which has a limited contribution to coherence or even partially counteracts it, thus the ergotropy in case (c) decreases and even becomes lower than case (b) with the increase of r.

It is worth noting that small deviations between the analytical and numerical results in the main text arise from the different approximations employed in the two approaches. In the numerical simulations, a finite-dimensional truncation of the Hilbert space is introduced, while the analytical treatment is obtained within a second-order mean-field approximation.

VI. SCATTERING COEFFICIENT MATRIX OF THE OPTICAL ISOLATOR

When both cavity modes are squeezed, the corresponding quantum Langevin equations (QLEs) can be found in Eq. (6) in the main text. To analyze the transmission properties, we linearize the QLEs as $dV/dt = -MV + DV_{\text{in}}$, where $V = (a_s, b_s, a_s^\dagger, b_s^\dagger)^T$ is the vector of system operators, $V_{\text{in}} = (a^{\text{in}}, b^{\text{in}}, a^{\text{in}\dagger}, b^{\text{in}\dagger})^T$, and $D = \text{diag}(\sqrt{\kappa_a}, \sqrt{\kappa_b}, \sqrt{\kappa_a}, \sqrt{\kappa_b})$. The coefficient matrix M can be written as

$$M = \begin{pmatrix} -\frac{\Lambda_a+i\omega_s}{2} & 0 & 0 & 0\\ 2J(e^{i\Delta\theta}\sinh r_a\sinh r_b-\cosh r_a\cosh r_b) & -\frac{\Lambda_b+i\omega_s}{2} & 2J(e^{-i\theta_a}\sinh r_a\cosh r_b-e^{-i\theta_b}\cosh r_a\sinh r_b) & 0\\ 0 & 0 & -\frac{\Lambda_a-i\omega_s}{2} & 0\\ 2J(e^{i\theta_a}\sinh r_a\cosh r_b-e^{i\theta_b}\cosh r_a\sinh r_b) & 0 & 2J(e^{-i\Delta\theta}\sinh r_a\sinh r_b-\cosh r_a\cosh r_b) & \frac{i\omega_s-\Lambda_b}{2}\end{pmatrix}.$$

Transforming to the frequency domain yields

$$\overline{V}(\omega) = (M - i\omega\mathbb{I})^{-1}D\overline{V}_{\text{in}}(\omega). \tag{S32}$$

Using the input-output relation $\nu_{\text{out}} + \nu_{\text{in}} = \sqrt{\gamma_\nu}\nu$ [S9], the output field is $\overline{V}_{\text{out}}(\omega) = U(\omega)\overline{V}_{\text{in}}(\omega)$, where

$$\overline{V}_{\text{out}} = \left(a^{\text{out}}(\omega), b^{\text{out}}(\omega), a^{\text{out}\dagger}(\omega), b^{\text{out}\dagger}(\omega)\right)^T, \tag{S33}$$

and

$$U(\omega) = D(M - i\omega\mathbb{I})^{-1}D - \mathbb{I}, \tag{S34}$$

which can be expressed as

$$\begin{pmatrix} -\frac{2\kappa_a}{\Lambda_a+i(2\omega+\omega_s)} & 0 & 0 & 0 \\ \frac{8J\sqrt{\kappa_a\kappa_b}(\cosh r_a\cosh r_b-e^{i\Delta\theta}\sinh r_a\sinh r_b)}{[\Lambda_a+i(2\omega+\omega_s)][\Lambda_b+i(2\omega+\omega_s)]} & \frac{-2\kappa_b}{\Lambda_b+i(2\omega+\omega_s)} & \frac{8J\sqrt{\kappa_a\kappa_b}(e^{-i\theta_b}\cosh r_a\sinh r_b-e^{-i\theta_a}\sinh r_a\cosh r_b)}{(i\Lambda_a-2\omega+\omega_s)(-i\Lambda_b+2\omega+\omega_s)} & 0 \\ 0 & 0 & -\frac{2\kappa_a}{\Lambda_a+i(2\omega-\omega_s)} & 0 \\ \frac{8J\sqrt{\kappa_a\kappa_b}(e^{i\theta_b}\cosh r_a\sinh r_b-e^{i\theta_a}\sinh r_a\cosh r_b)}{(i\Lambda_b-2\omega+\omega_s)(-i\Lambda_a+2\omega+\omega_s)} & 0 & \frac{8J\sqrt{\kappa_a\kappa_b}(\cosh r_a\cosh r_b-e^{-i\Delta\theta}\sinh r_a\sinh r_b)}{[\Lambda_a+i(2\omega-\omega_s)][\Lambda_b+i(2\omega-\omega_s)]} & \frac{-2\kappa_b}{\Lambda_b+i(2\omega-\omega_s)} \end{pmatrix} - \mathbb{I}. \tag{S35}$$

Substituting

$$\begin{aligned} a_s^{\rm out}(\omega) &= U_{11}(\omega)a_s^{\rm in}(\omega)+U_{12}(\omega)b^{\rm in}(\omega)+U_{13}(\omega)a_s^{\rm in\dagger}(\omega)+U_{14}(\omega)b^{\rm in\dagger}(\omega), \\ b^{\rm out}(\omega) &= U_{21}(\omega)a_s^{\rm in}(\omega)+U_{22}(\omega)b^{\rm in}(\omega)+U_{23}(\omega)a_s^{\rm in\dagger}(\omega)+U_{24}(\omega)b^{\rm in\dagger}(\omega), \end{aligned} \tag{S36}$$

and their Hermitian conjugate into the definition of the output field spectrum, $s_\nu^{\rm out}(\omega)=\int d\omega'\langle\overline{\nu}^{\rm out\dagger}(\omega')\overline{\nu}^{\rm out}(\omega)\rangle$ [S10], we can obtain

$$\begin{aligned} s_a^{\rm out}(\omega) &= \int d\omega' \sum_{j,k=1}^{4} U_{1j}^*(\omega')U_{1k}(\omega)\langle\overline{V}_{{\rm in},j}^\dagger(\omega')\overline{V}_{{\rm in},k}(\omega)\rangle, \\ s_b^{\rm out}(\omega) &= \int d\omega' \sum_{j,k=1}^{4} U_{2j}^*(\omega')U_{2k}(\omega)\langle\overline{V}_{{\rm in},j}^\dagger(\omega')\overline{V}_{{\rm in},k}(\omega)\rangle. \end{aligned} \tag{S37}$$

The subscript of $U(\omega)$ indicates its matrix element. Utilizing the correlation of the input field $\langle\nu^{\rm in\dagger}(\omega')\nu^{\rm in}(\omega)\rangle = s_\nu^{\rm in}(\omega)\delta(\omega-\omega')$, we can obtain

$$\begin{aligned} s_a^{\rm out}(\omega) &= \big(|U_{11}|^2+|U_{13}|^2\big)s_a^{\rm in}(\omega)+\big(|U_{12}|^2+|U_{14}|^2\big)s_b^{\rm in}(\omega)+|U_{13}|^2+|U_{14}|^2, \\ s_b^{\rm out}(\omega) &= \big(|U_{21}|^2+|U_{23}|^2\big)s_a^{\rm in}(\omega)+\big(|U_{22}|^2+|U_{24}|^2\big)s_b^{\rm in}(\omega)+|U_{23}|^2+|U_{24}|^2. \end{aligned} \tag{S38}$$

Furthermore, the spectrum of the output field can be expressed as

$$S_{\rm out}(\omega) = T(\omega)S_{\rm in}(\omega)+S_{\rm vac}(\omega), \tag{S39}$$

with $S_{\rm in}(\omega)=\big(s_a^{\rm in}(\omega),s_b^{\rm in}(\omega)\big)^T$, $S_{\rm out}(\omega)=\big(s_a^{\rm out}(\omega),s_b^{\rm out}(\omega)\big)^T$, $S_{\rm vac}(\omega)=\big(s_a^{\rm vac}(\omega),s_b^{\rm vac}(\omega)\big)^T$, and

$$T(\omega) = \begin{pmatrix} |U_{11}|^2+|U_{13}|^2 & |U_{12}|^2+|U_{14}|^2 \\ |U_{21}|^2+|U_{23}|^2 & |U_{22}|^2+|U_{24}|^2 \end{pmatrix}, \tag{S40}$$

$s_\nu^{\rm vac}$ is the output spectrum contributing from the input vacuum field, which is given by $s_a^{\rm vac}=|U_{13}|^2+|U_{14}|^2$ and $s_b^{\rm vac}=|U_{23}|^2+|U_{24}|^2$. With Eq. (S35), the transmission coefficient of the signal from mode b to mode a vanishes, $T_{ab}=0$, whereas the transmission coefficient from mode a to mode b is

$$\begin{aligned} T_{ba} =& \frac{64J^2\kappa_a\kappa_b}{\Lambda_b^2+(2\omega+\omega_s)^2}\left[\frac{2+\cosh\big[2(r_a-r_b)\big]+\cosh\big[2(r_a+r_b)\big]-2\cos\Delta\theta\sinh 2r_a\sinh 2r_b}{4\big[\Lambda_a^2+(2\omega+\omega_s)^2\big]}\right. \\ &\left.+\frac{\sinh^2 r_a\cosh^2 r_b+\cosh^2 r_a\sinh^2 r_b-\frac{1}{2}\cos\Delta\theta\sinh 2r_a\sinh 2r_b}{\Lambda_a^2+(-2\omega+\omega_s)^2}\right]. \end{aligned} \tag{S41}$$

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