nature portfolio

Peer Review File

The Riemann Hypothesis Manifested in Dynamical Quantum Phase Transitions

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This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors describe the emergence of the zeros of the Riemann function for two quantum systems coupled to a probe spin. A proof-of-principle set of experimental results is presented, together with a discussion of the complexity of the preparation of the initial state and of the time evolution under the Hamiltonian with eigenvalues the log of the natural.

I think the paper is interesting and presents valuable results, nevertheless there are several points that should addressed before the paper could be considered for publications. Let me go through the various points:

1) as far as I understand, in both schemes the hamiltonian {\cal H}_0=\sum_n log n |n><n| is used. In the first scheme the authors should clarify what is the total hamiltonian, i.e. how the probe spin is coupled to the system

2) I think an important point is that E_n should be having dimensions of an energy, and so 1/\beta. The authors should write for clarity first in dimensionful units and only after pass to dimensionless scheme (I suggest to do so also in the second scheme). The reason is that one can in this way appreciate why and whether \beta=1/2 is the point for the location of the zeros of the Riemann function

3) a very delicate point is that the authors keep going say that they are treating ``quantum many-body systems''. E.g., in the conclusion they say

``we have established a direct correspondence between the RH and the dynamics of quantum
many-body systems''

I disagree, or at least I think this is misleading. The hamiltonian {\cal H}_0 is the hamiltonian of a single-particle having a log spectrum, coupled with a probe spin. Why this is a ``strongly correlated system'' is not explained (and I think incorrect)

4) the log spectrum has been used in many papers, including C. Weiss et al.,
Physica A 341, 586 (2004). Since this is key point of the paper
the discussion of the references on this should be enlarged and made informative for the reader. Moreover, the literature on RH for physicists is rather extended, and I would add at least ``A Primer on the Riemann Hypothesis: The Celebration Collection'' by M. E. N. Tschaffon et al., together with ref. 15.

5) quite frankly, I do not agree with the title ``The Riemann Hypothesis Emerges in Dynamical Quantum Phase Transitions'' since it emerges for systems with a specific log spectrum and the authors fails to convey the idea that it is generic. Also the ``two'' systems seems to be the same system, after all: a log n system coupled with a probe, seen in two different ways.

Without an improuvement of the discussion of the previous points I think the paper cannot be considered for publication in Nature Communications.

Reviewer #2

(Remarks to the Author)

This paper studies the Riemann zeta function by relating it to measurable quantities of a time-evolved thermal state in a quantum system with energy levels E_n = log(n). The authors demonstrate that the zeros of the zeta functon(the Riemann hypothesis [RH] is that the nontrivial zeros have real part 1/2) can be located by an experimental realization of this system (plus a probe spin) on a small quantum computer. This seems to me to be an interesting application of quantum computing technology. However, my own expertise is in quantum many-body physics, rather than quantum computation, and so I do not know to what extent this should be regarded as a novel use of that technology.

Regarding the connection to quantum many-body physics, in my opinion the authors overstate the depth of this connection. In their concluding discussion, the authors write that “we have established a direct correspondence between RH and the dynamics of quantum many-body systems”. I would point out that the system in question, with energy levels E_n = log(n), is completely artificial, and does not correspond to any physically realizable quantum system. In particular, these energy levels imply a thermodynamic entropy (logarithm of the density of states) of S(E)=E exactly, which implies a constant temperature of unity for all energies, and hence infinite heat capacity. No such system exists, either in nature or in straightforward mathematical models of nature (eg systems of coupled spins). These energy levels are chosen because the partition function of the system is then closely related to the zeta function, but the lack of correspondence to any physically realizable system weakens the claim that there is a connection to “dynamics of quantum many body systems”.

Nevertheless I regard the paper as an interesting demonstration of the power of quantum computing technology, and therefore recommend publication.

Reviewer #3

(Remarks to the Author)
This peer-reviewed manuscript provides a fairly intensive study on the connection between number theory—specifically the Riemann Hypothesis (RH)—and quantum dynamics. Although the connection between the Riemann Hypothesis, or the Riemann zeta function, and quantum physics is known, this particular realization is new and original.

The manuscript is very well written. It is divided into theoretical and experimental parts, whose results are summarized in the main section of the article, while a more detailed description is provided at the end. The work is complemented by well-developed figures that contribute to understanding the overall problem, and the supplementary material is sufficient for a deeper understanding of the methods used.

The experimental verification of the theoretical predictions, gained using a specially tailored Hamiltonian that enables correspondence with the RH, greatly enriches the manuscript and strongly supports the evidence of an overlap between the RH and dynamical phase transitions. Moreover, the proposed quantum computational framework provides a bridge to quantum information.

Overall, this article opens up further possibilities for linking the Riemann Hypothesis with quantum physics, or more precisely, for presenting it as a natural part of it.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
I read the revised version and I think both the revised version and the reply addresses the points raised in my comments. I continue to rather disagree with the reference to "many-body" but I think the line of thought of the Authors

"Instead, we clarify that the “many-body” characterization refers to its implementation within a multi-qubit spin framework and the associated DQPT dynamics."

is rather satisfactory and not misleading. With the revised version I agree that the improuvement in clarity is substantial and I suggest publication.

Reviewer #2

(Remarks to the Author)
In my original review, I recommended publication, but noted some disagreements with some of the language the authors used to describe their results, finding it to overstate the breadth of the conclusions. Other referees had similar objections. In my view, the revised version has remedied all of these issues, and therefore I can now recommend publication with no reservations.

Dear Reviewers,

We gratefully acknowledge the valuable time and constructive comments you have dedicated to reviewing our manuscript. Your suggestions have significantly improved quality of our work. We have carefully considered all feedback and implemented the necessary modifications, which are highlighted in red within the updated manuscript. Additionally, we have prepared a detailed response addressing all the comments, which is included below.

Best regards,

Franco Nori and Guilu Long (on behalf of all the authors)

Summary of key changes:

[1] **Refining the Scope and Title:** Following the reviewers' and Editor's suggestions to ensure a more precise representation of our results, we have revised the title to "The Riemann Hypothesis Manifested in Dynamical Quantum Phase Transitions". Throughout the manuscript, we have moderated our claims and softened the presentation to accurately reflect the specific nature of the systems studied, avoiding any overstatement of generality.

[2] **Physical Units and Dimensional Consistency:** To enhance physical clarity as requested by Reviewer 1, we have introduced a dimensionful energy scale ε_0. The logarithmic spectrum is now initially presented in physical units, with the dimensionless parameters β and t rigorously defined thereafter. This consistent treatment has been applied to both proposed schemes to clarify the physical meaning of the critical dimensionless temperature.

[3] **Many-Body Representation and Scalability:** In particular, we no longer describe the model as a generic strongly correlated many-body system; instead, we clarify that it is an engineered spectral construction. At the same time, we added an explicit exact many-body spin representation for the finite-size logarithmic spectrum, and emphasized that the same target spectrum admits a polynomial-resource quantum-computational construction.

[4] **Expanded Literature and Theoretical Context:** We expanded the literature discussion by adding a broader overview of previous physics works related to the RH and logarithmic spectra, thereby placing our work in a clearer context.

Report of Referee 1

The authors describe the emergence of the zeros of the Riemann function for two quantum systems coupled to a probe spin. A proof-of-principle set of experimental results is presented, together with a discussion of the complexity of the preparation of the initial state and of the time evolution under the Hamiltonian with eigenvalues the log of the natural.

I think the paper is interesting and presents valuable results, nevertheless there are several points that should addressed before the paper could be considered for publications.

[Response:] We thank the referee for the careful reading and the positive assessment of our work.

[Comment 1:] as far as I understand, in both schemes the hamiltonian $\mathcal{H}_0 = \sum_n \log n|n\rangle\langle n|$ is used. In the first scheme the authors should clarify what is the total hamiltonian, i.e. how the probe spin is coupled to the system.

[Response:] We thank the referee for raising this point. In the first scheme, the total Hamiltonian can be viewed as a piecewise time-dependent Hamiltonian of dimension $2N \times 2N$,

$$\mathcal{H}_{\text{tot}} = |\downarrow\rangle\langle\downarrow| \otimes \begin{cases} \lambda(I - \sigma_z^1), & \text{in the first stage with duration } t_0, \\ \mathcal{H}_0, & \text{in the second stage with duration } t, \end{cases}$$

Specifically, after preparing the probe qubit in $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and the system in the thermal state ρ_s, we apply

$$\mathcal{H}_{c1} = \lambda\,|\downarrow\rangle\langle\downarrow| \otimes (I - \sigma_z^1)$$

for a duration t_0 with $\lambda t_0 = \pi/2$, followed by

$$\mathcal{H}_{c2} = |\downarrow\rangle\langle\downarrow| \otimes \mathcal{H}_0$$

for a duration t, while the $|\uparrow\rangle$ branch is left unchanged. The corresponding propagator is

$$U(t + t_0) = |\uparrow\rangle\langle\uparrow| \otimes I + |\downarrow\rangle\langle\downarrow| \otimes e^{-i\mathcal{H}_0 t}\sigma_z^1.$$

Therefore, the probe coherence is given by

$$\langle\sigma_x\rangle + i\langle\sigma_y\rangle = \mathrm{Tr}\left[\rho_s e^{-i\mathcal{H}_0 t}\sigma_z^1\right] = \mathcal{L}(\beta, t)$$

We have revised the manuscript to explicitly clarify the total Hamiltonian, specifically how the probe spin is coupled to the system.

[Changes:] We have made the corresponding updates in the revised manuscript as follows.

1. Clarification of the probe–system coupling in the first scheme in Section II A. To access $\mathcal{L}(\beta, t)$ experimentally, we introduce a probe spin prepared in the superposition state $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and couple it to the system prepared in the thermal state ρ_s. Specifically, the $|\downarrow\rangle$ branch of the probe first evolves under the Hamiltonian

$$\mathcal{H}_{c1} = \lambda |\downarrow\rangle \langle\downarrow| \otimes (I - \sigma_z^1)$$

for a duration t_0, satisfying $\lambda t_0 = \pi/2$. This imprints a phase factor of $(-1)^n$ onto the n-th component $|n\rangle \langle n|$ of ρ_s. The branch then evolves under

$$\mathcal{H}_{c2} = |\downarrow\rangle \langle\downarrow| \otimes \mathcal{H}_0$$

for a duration t, while the $|\uparrow\rangle$ branch remains unperturbed to serve as a reference. By measuring the probe coherence, we can directly extract

$$\langle\sigma_x\rangle + i\langle\sigma_y\rangle = \mathrm{Tr}\left[\rho_s e^{-i\mathcal{H}_0 t}\sigma_z^1\right] = \mathcal{L}(\beta, t),$$

thereby providing experimental access to the dynamical signatures.

[Comment 2:] I think an important point is that E_n should be having dimensions of an energy, and so $1/\beta$. The authors should write for clarity first in dimensionful units and only after pass to dimensionless scheme (I suggest to do so also in the second scheme). The reason is that one can in this way appreciate why and whether $\beta = 1/2$ is the point for the location of the zeros of the Riemann function.

[Response:] We thank the referee for this insightful suggestion. We agree that a dimensionful treatment clarifies the physical significance of the critical line at $\beta = 1/2$. Accordingly, we have revised the manuscript to include the explicit derivation from physical units to dimensionless variables for both schemes. Specifically, we introduce a characteristic energy scale ε_0 and express the energy levels as $E_n = \varepsilon_0 \log n$. This allows us to rigorously define the dimensionless variables:
(a) The dimensionless inverse temperature is $\beta = \frac{\varepsilon_0}{k_B T}$, where T is the physical temperature;
(b) The dimensionless time is $t = \varepsilon_0 t_{\mathrm{ph}}/\hbar$, where t_{ph} is the physical time.
Under this map, the partition function becomes $Z(T, \mathcal{H}_0^{\mathrm{ph}}) = \sum_{n=1}^{N} e^{-E_n/(k_B T)} = \sum_{n=1}^{N} n^{-\beta}$, and the evolution factor becomes $e^{-iE_n t_{\mathrm{ph}}/\hbar} = n^{-it}$, so the complex variable $s = \beta + it$ occurs naturally. Consequently, the critical line $\beta = 1/2$ now corresponds to a well-defined dimensionless inverse temperature.

We have also applied the same consistent dimensional analysis to the second scheme, ensuring that the logarithmic spectrum is first presented in dimensionful form before transitioning to the dimensionless representation. This makes the relation between the physical variables and the zeta-function variables more explicit.

[Changes:] We have made the corresponding updates in the revised manuscript as follows.

1. Clarification of the dimensionful Hamiltonian and the definitions of the dimensionless variables β and t in Section II A. We consider a quantum system with physical Hamiltonian
$$\mathcal{H}_0^{\mathrm{ph}} = \sum_{n=1}^{N} E_n \left|n\right\rangle \left\langle n\right|, \qquad E_n = \varepsilon_0 \log n,$$
where ε_0 sets the energy scale [Ref. 16: Mack *et al.*, Phys. Rev. A 82, 032119 (2010)]. The system is initialized in the thermal state $\rho_s = e^{-\mathcal{H}_0^{\mathrm{ph}}/(k_B T)}/\mathcal{Z}(T, \mathcal{H}_0^{\mathrm{ph}})$, with partition function
$$\mathcal{Z}(T, \mathcal{H}_0^{\mathrm{ph}}) = \sum_{n=1}^{N} e^{-E_n/(k_B T)} = \sum_{n=1}^{N} n^{-\varepsilon_0/(k_B T)}.$$
We then introduce the dimensionless inverse temperature, Hamiltonian, and time,
$$\beta \equiv \frac{\varepsilon_0}{k_B T}, \qquad \mathcal{H}_0 \equiv \frac{\mathcal{H}_0^{\mathrm{ph}}}{\varepsilon_0}, \qquad t \equiv \frac{\varepsilon_0 t_{\mathrm{ph}}}{\hbar}.$$
With these definitions, the thermal weight and dynamical phase of the n-th level become
$$e^{-E_n/(k_B T)} = n^{-\beta}, \qquad e^{-iE_n t_{\mathrm{ph}}/\hbar} = n^{-it},$$
and the partition function reduces to
$$\mathcal{Z}(\beta, \mathcal{H}_0) = \sum_{n=1}^{N} n^{-\beta}.$$
The complex parameter $s = \beta + it$ is therefore a natural dimensionless variable. In the following, energies and times are measured in units of ε_0 and $\hbar/\varepsilon_0$, respectively.
2. Clarification of the corresponding dimensionless notation in the second scheme in Section II B. As in the first scheme, we begin with the physical Hamiltonian $\mathcal{H}_0^{\mathrm{ph}}$ with logarithmic spectrum E_n defined in Equation (2), and adopt the same dimensionless variables as in Equation (4), namely the inverse temperature β, the rescaled Hamiltonian $\mathcal{H}_0$, and the time t.

[Comment 3:] A very delicate point is that the authors keep saying that they are treating "quantum many-body systems". E.g., in the conclusion they say "we have established a direct correspondence between the RH and the dynamics of quantum many-body systems". I disagree, or at least I think this is misleading. The hamiltonian $\mathcal{H}_0$ is the hamiltonian of a single-particle having a log spectrum, coupled with a probe spin. Why this is a "strongly correlated system" is not explained (and I think incorrect).

[Response:] We thank the referee for this critical observation. We agree that our previous wording that the diagonal Hamiltonian $\mathcal{H}_0 = \sum_n E_n|n\rangle\langle n|$ is inherently a natural strongly-correlated many-body system can be misleading and we have revised the manuscript accordingly. We appreciate the reviewer pointing out that such a spectrum can indeed be realized via an elegant single-particle construction [Ref. 16: Mack *et al.*, Phys. Rev. A 82, 032119 (2010)].

However, in the context of quantum computation, a spectrum is typically realized by mapping it onto the computational basis of multiple spins. To resolve any ambiguity and ensure consistency with the "many-body" terminology, we have added an explicit many-body spin representation in the revised manuscript. Specifically, by encoding the $N = 2^q$ energy levels into the basis of q qubits, the Hamiltonian can be expanded as a multi-spin model:

$$\mathcal{H}_0 = \sum_{l=0}^{q} \sum_{1\leq m_1<\cdots<m_l\leq q} J_{m_1\ldots m_l}\sigma^z_{m_1}\ldots\sigma^z_{m_l},$$

where the interaction coefficients $J_{m_1\ldots m_l}$ are uniquely determined by:

$$J_{m_1\ldots m_l} = \frac{1}{N}\sum_{n=1}^{N}(\log n) z^{(n)}_{m_1}\ldots z^{(n)}_{m_l}.$$

Here, $z^{(n)}_m = (-1)^{b^{(n)}_m}$ is determined by the binary representation of the index n. This representation demonstrates that the logarithmic spectrum can be exactly mapped onto a system of q interacting spins, involving high-order multi-body interactions. Although a direct implementation of such a many body spin model would typically require exponentially many interaction terms, our quantum computational framework overcomes this challenge. By allowing the construction of the target dynamics using only polynomial resources, our algorithm ensures the scalability of the method—a key contribution of this work.

We have revised the manuscript to moderate our language. We no longer describe $\mathcal{H}_0$ as an inherently strongly-correlated system. Instead, we clarify that the "many-body" characterization refers to its implementation within a multi-qubit spin framework and the associated DQPT dynamics. We hope that this clarification addresses the referee's concern regarding the physical interpretation of the system.

[Changes:] We have made the corresponding updates in the revised manuscript as follows.

1. Addition of an exact many-body spin representation of the finite-size logarithmic spectrum in Section II A. Equivalently, for finite N, the logarithmic spectrum admits an exact representation in terms of a many-body spin Hamiltonian. For $N = 2^q$ levels, encoding the

energy eigenstates into the computational basis of q spins yields

$$\mathcal{H}_0 = \sum_{l=0}^{q} \sum_{1 \leq m_1 < \cdots < m_l \leq q} J_{m_1 \cdots m_l}\, \sigma^z_{m_1} \cdots \sigma^z_{m_l}.$$

Writing $n-1 = \sum_{m=1}^{q} 2^{m-1} b_m^{(n)}$, with $b_m^{(n)} \in \{0,1\}$ and defining $z_m^{(n)} \equiv (-1)^{b_m^{(n)}}$, the coefficients are uniquely determined by

$$J_{m_1 \cdots m_l} = \frac{1}{N} \sum_{n=1}^{N} (\log n)\, z_{m_1}^{(n)} \cdots z_{m_l}^{(n)}, \ l = 0, 1, \ldots, q.$$

The logarithmic spectrum is therefore not tied to a unique microscopic realization, but admits an exact representation in a spin system.

2. Revision of the related wording in Section II D to avoid the interpretation of the model as a generic strongly correlated many-body system. Since the two dynamical constructions introduced above admit many-body spin representations with exponentially many interaction terms, their direct realization on a gate-based quantum computer is highly nontrivial.

[Comment 4:] The log spectrum has been used in many papers, including C. Weiss et al., Physica A 341, 586 (2004). Since this is key point of the paper the discussion of the references on this should be enlarged and made informative for the reader. Moreover, the literature on RH for physicists is rather extended, and I would add at least “A Primer on the Riemann Hypothesis: The Celebration Collection” by M. E. N. Tschaffon et al., together with ref. 15.

[Response:] We thank the reviewer for this helpful suggestion. We agree that, since the logarithmic spectrum is a central ingredient of our work, the related literature should be discussed more broadly and more informatively for the reader. We have therefore revised the Introduction and the discussion around the logarithmic spectrum in two respects.

First, we have expanded the discussion of earlier uses of logarithmic spectra in physics. In particular, we now mention their appearance in Bose–gas analogies for factorization [Ref. 27: C. Weiss *et al.*, Physica A 341, 586–606 (2004)], in wave-packet constructions related to the zeta function [Ref. 16: Mack *et al.*, Phys. Rev. A 82, 032119 (2010)], in factorization-oriented schemes based on interacting bosons or separable potentials [Ref. 28: F. Gleisberg *et al.*, New J. Phys. 15, 023037 (2013); Ref. 29: F. Gleisberg *et al.*, Phys. Lett. A 379, 2556–2560 (2015)], and in more recent holographic approaches [Ref. 30: D. Cassettari, G. Mussardo, and A. Trombettoni, PNAS Nexus 2, pgac279 (2023)]. This added discussion is intended to make clear that logarithmic spectra have already played a role in several number-theoretic and quantum settings.

Second, following the reviewer’s suggestion, we have also added reviews aimed at physicists [Ref. 5: D. Schumayer and D. A. W. Hutchinson, Rev. Mod. Phys. 83, 307–330 (2011); Ref. 10:

M. E. N. Tschaffon, I. Tkáčová, H. Maier, and W. P. Schleich, "A Primer on the Riemann Hypothesis," in Sketches of Physics, Lecture Notes in Physics 1000, 191–263 (2023)].

These additions place our work more clearly in the existing literature on logarithmic spectra and RH-related physics.

[Changes:] We have made the corresponding updates in the revised manuscript as follows.

1. Expansion of the discussion of previous works involving logarithmic spectra in Section II A. Such spectra have appeared previously in a variety of number-theoretic and quantum-dynamical settings, including Bose–gas analogies for factorization [Ref. 27: C. Weiss *et al.*, Physica A 341, 586–606 (2004)], wave-packet constructions related to the zeta function [Ref. 16: Mack *et al.*, Phys. Rev. A 82, 032119 (2010)], factorization-oriented schemes based on interacting bosons or separable potentials [Ref. 28: F. Gleisberg *et al.*, New J. Phys. 15, 023037 (2013); Ref. 29: F. Gleisberg *et al.*, Phys. Lett. A 379, 2556–2560 (2015)], and more recent holographic approaches [Ref. 30: D. Cassettari, G. Mussardo, and A. Trombettoni, Holographic realization of the prime number quantum potential, PNAS Nexus 2, pgac279 (2023)].
2. Addition of broader review references on the connection between the Riemann hypothesis and physics in the Introduction. and for signatures in diverse physical contexts [Ref. 5: D. Schumayer and D. A. W. Hutchinson, Rev. Mod. Phys. 83, 307–330 (2011); Ref. 10: M. E. N. Tschaffon *et al.*, A Primer on the Riemann Hypothesis, in Sketches of Physics: The Celebration Collection, Lecture Notes in Physics 1000, 191–263 (2023)], including relativistic amplitudes...

[Comment 5:] Quite frankly, I do not agree with the title "The Riemann Hypothesis Emerges in Dynamical Quantum Phase Transitions" since it emerges for systems with a specific log spectrum and the authors fails to convey the idea that it is generic. Also the "two" systems seems to be the same system, after all: a log n system coupled with a probe, seen in two different ways.

[Response:] We thank the referee for this critical and constructive assessment. This comment has helped us clarify the scope and presentation of the manuscript.

First, we agree that the term "Emerges" might inadvertently imply a universal occurrence across arbitrary systems. To better reflect the specific construction studied here where the RH mathematical structure is encoded in the dynamical critical behavior of a tailored quantum system, we have revised the title to ***"The Riemann Hypothesis Manifested in Dynamical Quantum Phase Transitions "***. This title is intended to more accurately reflect the scope of our results. We revised the wording accordingly throughout the manuscript.

Second, we acknowledge that both approaches utilize a probe qubit coupled to a system with a logarithmic spectrum. In the revised manuscript, we now explicitly describe them as "two complementary constructions within a framework" rather than two entirely separate physical systems. We also clarify that the two constructions are related yet distinct, and they differ in their observables, protocols, asymptotic regimes, and roles, as follows.

(a) Observables: The first construction is based on the average phase factor $\mathcal{L}(\beta, t)$, whereas the second is based on the generalized Loschmidt amplitude $\mathcal{G}(\beta, t)$.

(b) Protocols: The first employs sequential phase-imprinting and controlled-evolution scheme, while the second uses a distinct time-dependent Hamiltonian to generate Loschmidt dynamics.

(c) Asymptotic Regimes: The first construction connects to $\zeta(\beta + it)$ in the thermodynamic limit, whereas the second requires a joint thermodynamic and long-time limit.

(d) Roles: The first is used for experimental implementation, while the second is specifically designed for access to high-order zeros.

Third, in accordance with the reviewer's suggestion to avoid overstating generality, we have replaced "emergence" with "occurrence" in multiple places.

[Changes:] We have made the corresponding updates in the revised manuscript as follows.

1. Revision of the title to The Riemann Hypothesis Manifested in Dynamical Quantum Phase Transitions".
2. Revision of the description of the two systems as two complementary quantum systems within a framework.
3. Replaced "emerge/emergence" with "occur/occurrence", to better reflect the scope of our results.

   For instance, in the Abstract, the revised sentence reads: "This precise correspondence recasts the RH as the **occurrence** of phase transitions at a unique temperature..." Similarly, this change has been implemented in the Introduction and Section II to ensure a cautious tone.
4. Explicit clarification that the systems are engineered rather than generic physical many-body systems.

   In abstract: Here we establish a direct correspondence between these zeros and dynamical quantum phase transitions in two complementary **engineered** quantum many-body systems, characterized by the average accumulated phase factor and the Loschmidt amplitude, respectively.

   In discussion: In this work, we have established a direct correspondence between the RH and the dynamics of **engineered** quantum many-body systems, and have experimentally verified this connection on a physical platform.
5. Clarification of the difference between two systems in Section II C. Taken together, the two systems fit into a unified framework based on a probe qubit coupled to a logarithmic spectrum, while differing in observables, dynamical protocols, and asymptotic limits.

Report of Referee 2

This paper studies the Riemann zeta function by relating it to measurable quantities of a time-evolved thermal state in a quantum system with energy levels $E_n = \log(n)$. The authors demonstrate that the zeros of the zeta function (the Riemann hypothesis [RH] is that the nontrivial zeros have real part 1/2) can be located by an experimental realization of this system (plus a probe spin) on a small quantum computer. This seems to me to be an interesting application of quantum computing technology. However, my own expertise is in quantum many-body physics, rather than quantum computation, and so I do not know to what extent this should be regarded as a novel use of that technology.

[Response:] We thank the reviewer for this careful summary of our work and for the positive assessment that it represents an interesting application of quantum computing technology. We also appreciate the reviewer's perspective from quantum many-body physics, since one of our goals is precisely to build a bridge between nonequilibrium quantum dynamics and a problem in analytic number theory.

[Comment 1:] Regarding the connection to quantum many-body physics, in my opinion the authors overstate the depth of this connection. In their concluding discussion, the authors write that "we have established a direct correspondence between RH and the dynamics of quantum many-body systems". I would point out that the system in question, with energy levels $E_n = log(n)$, is completely artificial, and does not correspond to any physically realizable quantum system. In particular, these energy levels imply a thermodynamic entropy (logarithm of the density of states) of $S(E) = E$ exactly, which implies a constant temperature of unity for all energies, and hence infinite heat capacity. No such system exists, either in nature or in straightforward mathematical models of nature (eg systems of coupled spins). These energy levels are chosen because the partition function of the system is then closely related to the zeta function, but the lack of correspondence to any physically realizable system weakens the claim that there is a connection to "dynamics of quantum many body systems". Nevertheless I regard the paper as an interesting demonstration of the power of quantum computing technology, and therefore recommend publication.

[Response:] We sincerely thank the reviewer for the careful reading, constructive criticism, and recommendation for publication. The reviewer raises one main concern, which we address in two parts below.

(a) Clarification of the physical interpretation and wording

We acknowledge that our original language overstates the generality of the physical connection. The reviewer is correct that the logarithmic spectrum $E_n = \log(n)$ does not arise in natural condensed-matter systems, and carries the unusual thermodynamic property of a constant temperature $T = 1$ for all energies. The spectrum is chosen to encode the arithmetic structure of the Riemann zeta function into the partition function. We agree with the reviewer's assessment and have revised the relevant sentence in the Discussion accordingly. The original statement "we have established a direct correspondence between the RH and the dynamics of quantum many-body systems" now reads:

> "we have established a direct correspondence between the RH and the dynamics of **engineered quantum many-body systems**, and have experimentally verified this connection on a physical platform."

We have also added a clarifying remark in Section II A, when $\mathcal{H}_0$ is first introduced, explicitly noting that the logarithmic spectrum is engineered to encode the structure of the zeta function rather than arising from a naturally occurring physical model.

(b) Engineered quantum systems and the many-body spin representation

At the same time, we note that the logarithmic spectrum $E_n = \log(n)$ has appeared previously in a variety of number-theoretic and quantum-dynamical settings, including Bose–gas analogies for factorization [Ref. 27: C. Weiss *et al.*, Physica A 341, 586–606 (2004)], wave-packet constructions related to the zeta function [Ref. 16: Mack *et al.*, Phys. Rev. A 82, 032119 (2010)], factorization-oriented schemes based on interacting bosons or separable potentials [Ref. 28: F. Gleisberg *et al.*, New J. Phys. 15, 023037 (2013); Ref. 29: F. Gleisberg *et al.*, Phys. Lett. A 379, 2556–2560 (2015)], and more recent holographic approaches [Ref. 30: D. Cassettari, G. Mussardo, and A. Trombettoni, Holographic realization of the prime number quantum potential, PNAS Nexus 2, pgac279 (2023)]. In most of these works, the logarithmic spectrum is generated through analog constructions, typically by designing specific potentials or effective physical landscapes.

A focus of our work is the development of a special-purpose quantum simulation framework. A central strength of quantum computation is the ability to realize "synthetic" or "artificial" Hamiltonians that do not exist in nature. Our system is not intended as a model of a natural material or a standard coupled-spin chain; it is an **engineered quantum many-body system** designed specifically to map a hard problem in number theory onto controllable quantum dynamics.

To resolve any remaining ambiguity regarding the "many-body" character of the system, we have added an explicit many-body spin representation to the revised manuscript. By encoding the $N = 2^q$ energy levels into the computational basis of q qubits, the Hamiltonian $\mathcal{H}_0$ admits an exact expansion as a multi-spin model:

$$\mathcal{H}_0 = \sum_{l=0}^{q} \sum_{1\leq m_1<\cdots<m_l\leq q} J_{m_1\cdots m_l}\, \sigma^z_{m_1}\cdots\sigma^z_{m_l}, \tag{1}$$

where the interaction coefficients are uniquely determined by:

$$J_{m_1\cdots m_l} = \frac{1}{N}\sum_{n=1}^{N}(\log n)\, z^{(n)}_{m_1}\cdots z^{(n)}_{m_l}, \tag{2}$$

with $z^{(n)}_m = (-1)^{b^{(n)}_m}$ determined by the binary representation of the index n. This representation demonstrates that the logarithmic spectrum corresponds exactly to a system of q interacting spins with higher-order many-body interactions.

We further note that a naive direct implementation of this many-body spin model would require exponentially many interaction terms. A key technical contribution of our work is the quantum computational framework that constructs the target dynamics using only polynomial resources, thereby ensuring scalability (details in Section II D). This highlights the role of quantum computation in simulating target dynamics that would otherwise involve exponentially many interaction terms in the direct many-body spin representation.

We hope that these revisions, namely the refined wording emphasizing the **engineered quantum many-body system** and the addition of the explicit many-body spin representation, fully address the reviewer's concern, and we are grateful for the feedback that prompted this clarification.

[Changes:] We have made the corresponding updates in the revised manuscript as follows.

1. Revision of the title to "The Riemann Hypothesis Manifested in Dynamical Quantum Phase Transitions".
2. We have substituted "emerge/emergence" with more restrained formulations, such as "occur/occurrence" to more accurately reflect the scope of our results.
   For instance, in the Abstract, the revised sentence reads: "This precise correspondence recasts the RH as the **occurrence** of phase transitions at a unique temperature..." Similarly, this change has been implemented in the Introduction and Section II to ensure a cautious and rigorous tone.
3. Explicit clarification that the systems are engineered rather than generic physical many-body systems.
   In abstract: Here we establish a direct correspondence between these zeros and dynamical quantum phase transitions in two complementary **engineered** quantum many-body systems, characterized by the average accumulated phase factor and the Loschmidt amplitude, respectively.

In discussion: In this work, we have established a direct correspondence between the RH and the dynamics of **engineered** quantum many-body systems, and have experimentally verified this connection on a physical platform.

4. Expansion of the discussion of engineered realizations and related precedents in Section II A. Such spectra have appeared previously in a variety of number-theoretic and quantum-dynamical settings, including Bose–gas analogies for factorization [Ref. 27: C. Weiss *et al.*, Physica A 341, 586–606 (2004)], wave-packet constructions related to the zeta function [Ref. 16: Mack *et al.*, Phys. Rev. A 82, 032119 (2010)], factorization-oriented schemes based on interacting bosons or separable potentials [Ref. 28: F. Gleisberg *et al.*, New J. Phys. 15, 023037 (2013); Ref. 29: F. Gleisberg *et al.*, Phys. Lett. A 379, 2556–2560 (2015)], and more recent holographic approaches [Ref. 30: D. Cassettari, G. Mussardo, and A. Trombettoni, Holographic realization of the prime number quantum potential, PNAS Nexus 2, pgac279 (2023)].
5. Addition of an exact many-body spin representation of the finite-size system in Section II A. Equivalently, for finite N, the logarithmic spectrum admits an exact representation in terms of a many-body spin Hamiltonian. For $N = 2^q$ levels, encoding the energy eigenstates into the computational basis of q spins yields

$$\mathcal{H}_0 = \sum_{l=0}^{q} \sum_{1\leq m_1<\cdots<m_l\leq q} J_{m_1\cdots m_l}\, \sigma^z_{m_1} \cdots \sigma^z_{m_l}.$$

Writing $n-1 = \sum_{m=1}^{q} 2^{m-1} b_m^{(n)}$, with $b_m^{(n)} \in \{0,1\}$ and defining $z_m^{(n)} \equiv (-1)^{b_m^{(n)}}$, the coefficients are uniquely determined by

$$J_{m_1\cdots m_l} = \frac{1}{N}\sum_{n=1}^{N} (\log n)\, z_{m_1}^{(n)} \cdots z_{m_l}^{(n)},\ l = 0,1,\ldots,q.$$

The logarithmic spectrum is therefore not tied to a unique microscopic realization, but admits an exact representation in a spin system.

Report of Referee 3

This peer-reviewed manuscript provides a fairly intensive study on the connection between number theory—specifically the Riemann Hypothesis (RH)—and quantum dynamics. Although the connection between the Riemann Hypothesis, or the Riemann zeta function, and quantum physics is known, this particular realization is new and original.

The manuscript is very well written. It is divided into theoretical and experimental parts, whose results are summarized in the main section of the article, while a more detailed description is provided at the end. The work is complemented by well-developed figures that contribute to understanding the overall problem, and the supplementary material is sufficient for a deeper understanding of the methods used.

The experimental verification of the theoretical predictions, gained using a specially tailored Hamiltonian that enables correspondence with the RH, greatly enriches the manuscript and strongly supports the evidence of an overlap between the RH and dynamical phase transitions. Moreover, the proposed quantum computational framework provides a bridge to quantum information.

Overall, this article opens up further possibilities for linking the Riemann Hypothesis with quantum physics, or more precisely, for presenting it as a natural part of it.

[Response:] We sincerely thank the reviewer for this very positive and encouraging assessment of our work. We are grateful for the reviewer's recognition of the originality of this realization, the clarity of the presentation, the quality of the figures and supplementary material, and the value of combining the theoretical analysis with experimental verification and a quantum computational framework. We are also encouraged by the reviewer's view that this work may open further possibilities for connecting the Riemann Hypothesis with quantum physics.